POTENTIAL TOPOGRAPHY AND MASS GENERATION

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INTRODUCTION

We describe an approach to understanding exponential decay of correlation functions in asymptotically free theories. This approach is systematic; it does not start from any conjectured mechanism or picture. We begin by studying

- the metric on the space of configurations and

- the behavior of the potential-energy function on this space.

We begin by describing how these ideas fit in the framework of QCD, as discussed earlier [1]. We then consider the 1+1-dimensional $O(2)$ and $O(3)$ nonlinear sigma models and show that no gap exists in the former at weak coupling. In the $O(3)$ model a new kind of strong/weak-coupling duality is realized. We then briefly outline our proposals for understanding the spectrum.

THE YANG-MILLS METRIC

In the last few decades, there have been many serious attempts to understand

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wave functionals on orbit space by isolating a fundamental region (the interior of the Gribov horizon) [2]. Instead we examine this space using "automorphic functions", i.e. gauge-invariant wave functionals [1,3]. An interesting approach along similar lines for $2 + 1$-dimensional gauge theories is that of reference 4.

The Hamiltonian of the $D + 1$-dimensional $SU(N)$ Yang-Mills theory in $A_0 = 0$ (temporal) gauge

$$H = \int_M d^D x \, [-\frac{e_0^2}{2} \, tr \, \frac{\delta^2}{\delta \, A_j(x)^2} + \frac{1}{4e_0^2} \, tr \, F_{j\,k}(x)^2] \, .$$

The allowed wave functionals Ψ satisfy the condition that if A and B are physically equivalent (same up to gauge transformations of C-S number zero) $\Psi[A] = \Psi[B]$.

This will be reformulated as a particle on a certain infinite-dimensional curved space on which there is a height function, namely the potential energy. We are interested ultimately in how geodesic motion (strong couping) is influenced by this height function (potential topography).

The lattice discussion will be used here to introduce the Yang-Mills metric. A more heuristic motivation following Feynman [5] was used in reference 1.

Consider a lattice gauge theory with D space and 1 time dimension. Label discrete points in Euclidean space-time by x and t, respectively. Let $\{U(t)\}$ denote the set of lattice gauge fields (in the fundamental representation of $SU(N)$) on links pointing in space directions at a particular time . Split the action S into a space-time plaquette term S_{st} and a space-space plaquette term S_{ss}, i.e. $S = S_{st} + S_{ss}$, where

$$S_{st} = -\sum_t \frac{1}{2e_0^2} \Re[\{U(t)\}, \{U(t+1)\}]^2 \, , \quad S_{ss} = -\sum_t \frac{1}{4e_0^2} \ell[\{U(t)\}] \, ,$$

where

$$\Re[\{U(t)\}, \{U(t+1)\}]^2 = \frac{1}{2} \sum_{x,i} tr[U_i(x,t)U_0(x+\hat{i},t)U_i(x+\hat{t},t)^\dagger U_0(x,t)^\dagger] + c.c. \, ,$$

and

$$\ell[\{U(t)\}] = \sum_{x,i<j} tr[U_i(x,t)U_j(x+\hat{i},t)U_i(x+\hat{j},t)^\dagger U_j(x,t)^\dagger] + c.c. \, .$$

Now let's try to integrate out the links pointing in time direction, $U_0(x,t)$. As a first approximation to doing this, we can just solve their equation of motion. This says that $\Re[\{U(t)\}, \{U(t+1)\}]^2$ is minimized with respect to these degrees of freedom. If we integrate them out explicitly the result is a product of a Bessel functions. Near the maximum of this product it has the form $\exp - \sum_t \rho[\{U(t)\}, \{U(t+1)\}]$, where ρ is the *absolute* (not local) minimum

$$\rho[\{U(t)\}, \{U(t+1)\}] = \min_{\{U_0(t)\}} \Re[\{U(t)\}, \{U(t+1)\}] \, .$$

The quantity $\rho[\{U(t)\}, \{U(t+1)\}]$ can be shown to be a metric on the space $\{U(t)\}$ modulo gauge transformations. Thus the kinetic term in the action by itself describes Brownian motion in this space.

In the continuum the space of connections $\mathcal{U}$ is defined to contain only those gauge fields which are Lebesgue measurable, and square-integrable [1]. No distinction is made between gauge fields which are the same almost everywhere ($\mathcal{U}$ is a Hilbert space). Gauge transformations are $SU(N)$ valued functions $g(x)$ which are differentiable and for which $ig^{-1}\partial g \in \mathcal{U}$. Any element of $\mathcal{U}$ is mapped into another element of $\mathcal{U}$ by such a gauge transformation.

The equivalence classes must actually be made larger in order to obtain a metric space $\mathcal{M}_D$. Two vectors in $\mathcal{U}$ with representatives A and B will be said to be gauge-equivalent if there is a sequence of gauge transformations g_1, g_2,..., such that

$$B = \lim_n A^{g_n}$$

in the usual metric of the Hilbert space $\mathcal{U}$.

Let α and β be two physical configurations, with A a representative of α and B a representative of β. The distance function [5, 6] is defined by

$$\rho[\alpha,\beta] = \inf_{h,f}\{\sqrt{\frac{1}{2}\int_M d^D x \; tr \; [A^h(x) - B^f(x)]^2}\} \; .$$

This is just the continuum version of the lattice metric defined above. The function $\rho[\cdot,\cdot]$ was shown to give a complete metric space on equivalence classes of gauge connections $\mathcal{M}_D$.

There is a local metric on the space of connections. This turns out to be essentially that discussed some time ago [7, 8]. The Laplacian actually contains several terms not found by these authors [1, 9]. The geodesics of the space can be proved to be those conjectured by Babelon and Viallet [8]. With the metric tensor defined properly, there are no non-generic points as had been claimed (the orbit space is complete). We believe the problem some people found with non-generic points is related to the fact that they worked with connections in Sobolev space rather than those in $\mathcal{U}$ in which case there is no longer completenss using the metric $\rho[\cdot,\cdot]$.

The square of the infinitesmal distance in orbit space $\mathcal{M}_D$ due to a small displacement δA in $\mathcal{U}$ is

$$d\rho^2 = [\int_M d^D x \sum_{j=1}^{D} \sum_{a=1}^{N^2-1}] [\int_M d^D y \sum_{k=1}^{D} \sum_{b=1}^{N^2-1}] \; G_{(x,j,a)\;(y,k,b)} \; \delta A_j^a(x) \delta A_k^b(y) \; ,$$

where the metric tensor is

$$G_{(x,j,a)\;(y,k,b)} = \delta_{j\;k}\delta_{a\;b}\delta^D(x-y) - (\mathcal{D}_j \; \mathcal{P}\frac{1}{\mathcal{D}^2} \; \mathcal{D}_k)_{a\;b} \; \delta^D(x-y) \; ,$$

and where in the Green's function $\mathcal{P}\frac{1}{\mathcal{D}^2}$, the principle value projects out the zero modes of $\mathcal{D}^2$. Reducible connections [10] are a set of measure zero.

This metric tensor has zero eigenvalues; the dimension of the coordinate space $\mathcal{U}$ is larger than the dimension of the orbit space. One must either gauge fix (and deal with the Gribov problem by prescribing a fundamental domain) or develop methods for

Riemannian geometry for metric tensors with zero modes. We follow the latter path. The Laplacian was first found by Schwinger on the basis of relativistic invariance and further discussed by Gawedzki [9]. It was constructed in reference 1 using a theory of tensors applicable when the dimension of coordinate space is greater than the dimension of the manifold. It is

$$\Delta \Psi[\alpha] = -\partial_Y (G^{YU} \partial_U \Psi[\alpha]) + (\partial_Z G_Y^Z) G^{YU} \partial_U \Psi[\alpha] \, ,$$

where capital Roman letters denote "indices" $X = (x, j, a)$ and $\partial_X = \frac{\delta}{\delta A_i^a(x)}$.

The Yang-Mills Hamiltonian is

$$H = \frac{e^2}{2} \Delta + \frac{e^2 \mathcal{R}}{12} + \int_M d^D x \, \frac{1}{4e^2} \, tr \, F_{jk}(x)^2 \, ,$$

where $\mathcal{R}$ denotes the ultraviolet divergent scalar curvature.

STRUCTURE OF THE YANG-MILLS POTENTIAL ENERGY - RIVER VALLEYS AND GLUONS

The metric properties of the manifold $\mathcal{M}_D$ of configurations determine the spectrum of the kinetic term of the Hamiltonian. To understand the spectrum at weak coupling, the potential or magnetic energy must be examined.

A natural starting point is to try make a relief map of magnetic energy on $\mathcal{M}_D$; in other words to investigate the magnetic topography [1].

Suppose that the manifold of physical space is very large. Make a rescaling of the coordinates and the connection $A \in \mathcal{U}$ by a real factor s:

$$A_j(x) \longrightarrow s A_j(sx) \, .$$

A gauge-transformed connection A^h will transform the same way under a rescaling, provided $h(x)$ is redefined by

$$h(x) \longrightarrow h(sx) \, .$$

The distance of the point of orbit space α from an equivalence class of pure gauges, α_0, transforms as

$$\rho[\alpha, \alpha_0] \longrightarrow s^{\frac{2-D}{2}} \rho[\alpha, \alpha_0] \, .$$

Let $A \in \mathcal{U}$ be a particular configuration of finite potential energy, for which the magnetic field $F_{jk}(x)$ decays rapidly to zero for x outside some finite bounded region, which will be called the **domain** of the magnetic field. By changing the size of the domain and the magnitude of the magnetic field, the distance from some given pure gauge can be made arbitrarily small (except when regularization effects become important) or large (except when volume effects become important).

The potential energy

$$U[\alpha] = \int_M d^D x \, \frac{1}{4e^2} \, tr \, F_{jk}^2(x) \, ,$$

transforms as

$$U[\alpha] \longrightarrow s^{4-D} U[\alpha]$$

and so for $D > 2$

$$U \sim \rho^{-2\frac{4-D}{D-2}} \, .$$

For $2 < D < 4$ the exponent is negative. Thus it is possible to have arbitrarily large U for arbitrarily small ρ.

For Abelian gauge theories, other rescalings can be considered;

$$A_j(x) \longrightarrow s^\phi A_j(sx) \, ,$$

where ϕ is any real number. By choosing ϕ satisfying $\frac{D-2}{2} < \phi < \frac{D}{2}$ it is always possible to make the potential energy arbitrarily small for small s, no matter what the dimension. The domain can be made large, while the field strength is made small; a quantum wave functional $\Phi[\alpha]$ whose amplitude is largest near this configuration is a long-wavelength photon. This quantum state must be orthogonal to the vacuum $\Psi_0[\alpha]$ because at least one of the two wave functionals is zero at any point in orbit space. This is why non-compact electrodynamics has no mass gap in any dimension. Our analysis seems to indicate that the same is true for Yang-Mills theory for dimension between $2+1$ and $4+1$.

Figure 1. illustrates the situation. Orbit space contains regions, which we call **river valleys** in which the potential energy vanishes in the thermodynamic limit. The configurations in the river valleys are not pure-gauge configurations α_0. The river valleys are preserved under scale transformations and are therefore finite-dimensional. In perturbation theory only the region near α_0 is explored. Perturbative gluons are oscillations along straight line extrapolations of these curves.

OUR STRATEGY

In the light of the above discussion, how could the spectrum of QCD possibly contain anything other that massless particles?

Answer: The regions of small potential energy could have large electric (kinetic) energy by the uncertainty principle. The zero-point energy of the modes transverse to the river valleys must be added to the potential. In this way, the first excited state could have a finite gap above the ground state. There are examples of quantum-mechanical systems, including models motivated by Yang-Mills theory [11], for which this is true [12].

We should view the position along each river valley as a collective coordinate. We then integrate out all degrees of freedom except this coordinate. The resulting quantum-mechanical Hamiltonian will have eigenstates which correspond to the eigenstates of
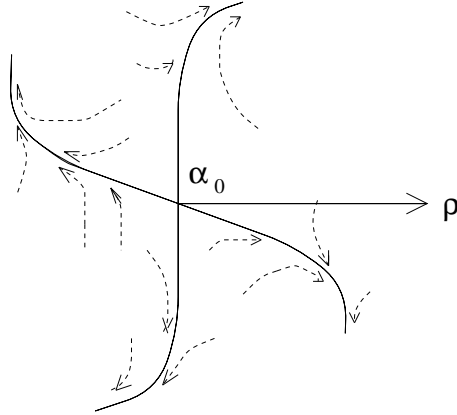
Figure 1. The topography of the Yang-Mills theory in four space-time dimensions. The dashed arrows represent directions of decreasing potential energy along scale transformations. The radius ρ is the distance from the zero potential configuration α_0. The solid curves (river valleys) are where the potential energy vanishes in the thermodynamic limit (only a finite subset is depicted).

the field-theory Hamiltonian with zero total momentum. To insure consistency of the collective-coordinate approximation, we will must consider only small fluctuations around river valleys.

THE $O(n)$ NONLINEAR SIGMA MODEL

A great deal is known about the $1 + 1$-dimensional sigma models. The phase transition in the $O(2)$ model is well understood [13]. By virtue of its integrability [14] the S-matrix [15] and spectrum [16] of the $O(3)$ model are also known. Unfortunately neither these methods nor the $1/n$-expansion [17] extend to gauge theories in higher dimensions.

This model will first be considered in $D + 1$ dimensions. Later we will specialize to $D = 1$. The field $s(x)$ (we are fixing time) with x on a D-dimensional lattice is a real n-tuple with $s^T(x)s(x) = 1$. The Hamiltonian is

$$H = \frac{e_0}{2} \sum_x L(x)^T L(x) - \frac{1}{2e_0} \sum_{<x,x'>} s(x)^T s(x') .$$

The fields s lie in equivalence classes:

$$\psi = \{Rs; \ R \in O(n)\}.$$

The definition of these equivalence classes isn't yet obviously right. Unlike the case of Yang-Mills theory, the equivalence class contains physically *different* configurations. We will worry about this issue later.

A natural metric on equivalence classes ψ, ϕ with $s \in \psi$, $f \in \phi$ is [5]

$$\rho[\phi, \psi]^2 = \inf_{R \in O(n)} \sum_x [Rf(x) - s(x)]^T [Rf(x) - s(x)] .$$

In the continuum this goes over to (up to factors of the lattice spacing)

$$\rho[\phi, \psi]^2 = \inf_{R \in O(n)} \int d^D x \, [Rf(x) - s(x)]^T [Rf(x) - s(x)] \tag{1}$$
$$= V - tr\sqrt{M^T M} ,$$

where V is now the volume of the space manifold and

$$M_{jk} = \int d^D x \, f_j(x) s_k(x) .$$

However, we no longer strictly have a metric without making certain restrictions on allowed spin configurations. Without such restrictions, different configurations are separated by a distance zero. This is a minor difficulty and will not trouble us.

THE SIGMA-MODEL RIVER VALLEYS

Let's denote the "pure gauge" configuration containing constant $s(x)$ by ψ_0.

Consider now the following problem for $D = 1$. For fixed $\rho[\psi, \psi_0]$ minimize the the potential energy

$$U(\psi) = \int_0^L dx \, s'(x)^2 , \quad s \in \psi .$$

subject to Neumann boundary conditions $s'(0) = s'(L) = 0$. Let's parametrize $s(x)$ using angles $\theta_1(x),..., \theta_{n-1}(x)$, by $s_1(x) = \sin\theta_1(x)... \sin\theta_{n-1}(x)$, ..., $s_n = \cos\theta_1(x)$, in the standard way.

The solution for the minima of $U(\psi)$ for fixed $\rho = \rho[\psi, \psi_0]$ (distance from the origin=pure gauge) is similar to that of a pendulum. We find that up to global rotations R there are minima labeled by an integer $N = 1, 2, ...$

$$\theta_1(x) = \pm\alpha_N(x, k) = \pm 2\sin^{-1} k \, sn(\frac{2NK}{L}x - K) ,$$

$$\theta_2(x) = \cdots = \theta_{n-1}(x) = 0 ,$$

where $0 \le k \le 1$ is the modulus of the elliptic function $sn(u)$ and $K = K(k) = sn^{-1}(1)$ is the usual complete elliptic integral. The river valleys are nicely parametrized by the modulus k as shown in figure 2.

We find that

$$\rho(k)^2 = \begin{cases} 2L(1 - \frac{E}{K}) , & 0 \le k \le k^* \approx 0.82 \\ 2L\frac{E}{K} , & k^* \le k \le 1 \end{cases} , \tag{2}$$

where $E = E(k) = \int_0^1 dn^2(u)\,du$ is another standard elliptic integral. This function rises smoothly from 0 to L as k goes from 0 to k^*, then falls off to zero again as $k \to 1$. In fact, on the lattice the $k = 1$ solution is unphysical, because this solution
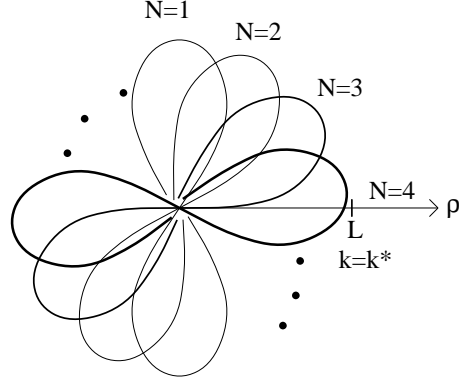
Figure 2. The river valleys for the $O(n)$ sigma model. The potential energy is nearly constant along the solid curves. As before, a straight line extrapolation along tangent vectors at the origin gives the spin wave approximation.

has discontinuities in the continuum. Actually $k \leq k_{max} \approx 1$ because of the regulator. A configuration along a river valley is maximally far from the origin at $k = k^*$.

The potential energy function is

$$U(k) = \frac{32N^2 K}{L} [E - (1 - k^2)K] .$$

For fixed volume L this diverges at $k = 1$, but, as mentioned earlier, this divergence is regularized by a lattice (or some other ultraviolet cut-off).

In the infinite volume limit for $k < k^*$, the potential is a constant; but the one-dimensional domain over which this is so has an infinite length ($=L$). If we view k as a collective variable, and ignore fluctuations in other degrees of freedom the gap is $O(\frac{1}{L})$.

We note that the river valleys are not straight lines in configuration space. Their tangent vectors at k are

$$\beta_N(x, k) = \frac{\partial \alpha_N(x, k)}{\partial k} = \frac{2sn(u)dn(u) - Z(u)cn(u)}{1 - k^2} ,$$

where $u = 2NKx/L - K$ and $Z(u)$ is the Jacobi zeta function. The inner product of β_N and its derivative with respect to k is not zero; this means that the river valleys are curved. The tangent vector does not have unit length in our collective coordinates. It is convenient to define the unit tangent vector $\hat{\beta}_N(x, k) = \beta_N(x, k)/\sqrt{\int_0^L \beta_N(y, k)^2 \, dy}$.

COLLECTIVE COORDINATES

Up to now we've ignored the fact that a system with global symmetry has states which transform as some representation of that symmetry (For Yang-Mills theories, we have no such problem). For example if $n = 2$ our river valleys are not one-dimensional, but two-dimensional surfaces parametrized by θ_0 as well as k: $\theta(x) = \theta_0 \pm \alpha_N(x, k)$. In fact the river valleys of the $O(n)$ sigma model are really $n - 1$-dimensional manifolds. However this consideration is irrelevant if we are asking for only certain information.

The degree of freedom corresponding to θ_0 is the very longest wavelength Goldstone mode. We can remove this mode if we are interested in mass spectra only and don't care about degeneracies of our states. For example, this can be done in the $O(3)$ model by adding a term $-\int dx dt \, \frac{\lambda}{V} s_1^2/(1-s_2^2)$, where V is the space-time volume. Such a term is of no consequence in the thermodynamic limit but clearly keeps the river valleys one-dimensional, rather than three-dimensional manifolds. This won't matter as long as we aren't interested in the transformation properties or non-accidental degeneracies of our states.

The collective-coordinate representation [18] of $\theta_1(x)$ is

$$\theta_1(x,t) = \alpha_N(x, k(t)) + \sum_{a=1}^{\infty} w_a(t) T_N^a(x, k(t)) \, ,$$

where we have now taken the range of $k(t)$ to be $-k_{max} < k(t) < k_{max}$ and $\alpha_N(x, -k) \equiv -\alpha_N(x, k)$. The family of functions $T_N^a(x, k)$ satisfy

$$\int_0^L T_N^a(x,k) T_N^b(x,k) \, dx = \delta^{ab} \, , \quad \int_0^L T_N^a(x,k) \hat{\beta}_N(x,k) \, dx = 0 \, ,$$

and $T^{a\,\prime}(0) = T^{a\,\prime}(L) = 0$.

Physically, the collective coordinate k is the parameter along the Nth river valley, while the $w_a(x)$'s are coordinates of dispacements normal to the valley at the point described by k.

In order to proceed further, it is necessary to compute the Jacobian $\|J\|$ of the transformation to the collective coordinate system. The details of this computation will be presented elsewhere. The answer turns out to be

$$\|J\| = \sqrt{\int_0^L \beta_N(x,k)^2 \, dx} - \sum_{a=1}^{\infty} w_a \int_0^L \frac{\partial \hat{\beta}_N(x,k)}{\partial k} T^a(x,k) \, dx \, .$$

NO GAP FOR THE $O(2)$ SIGMA MODEL

The behavior of the $O(2)$ sigma model at weak coupling is generally regarded as obvious [13]. From our perspective however, this model is nearly as hard to understand as all the others.

The functional measure on $\theta_1 = \theta$ is fairly simple. The fact that it is compact is responsible for vortices which are transitions between river valleys We will ignore these transitions for the following reason. A vortex can be regarded as a process through which a configuration evolves in time; the initial and final configurations are in the same equivalence class. Thus it is a loop in the metric space. The length of this loop can be computed and diverges in the thermodynamic limit. In the spirit of our approach, which is to consider only small fluctuations near river valleys, vortices can therefore be ignored. The details of this elementary calculation will be presented elsewhere.

The only nontrivial factor in the measure is the Jacobian to collective coordinates. The path integral

$$Z = \int \mathcal{D}\theta(x,t) \exp - \int dt \int_0^L dx \, \frac{1}{2e_0} (\partial_t \theta^2 + \partial_x \theta^2)$$
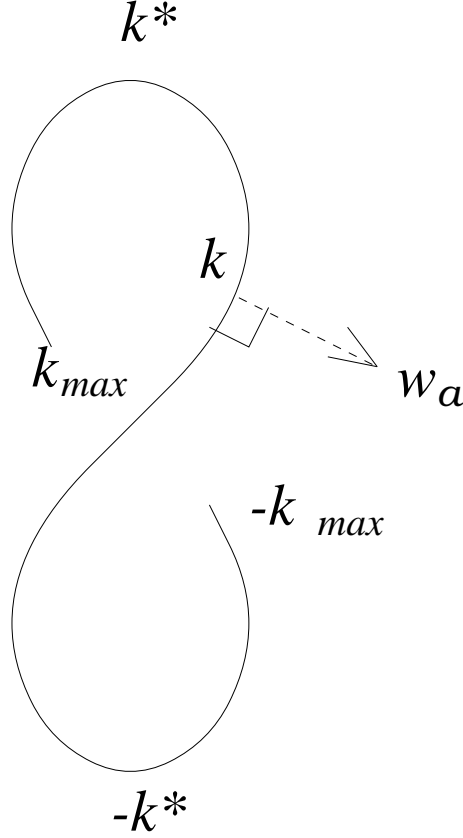
Figure 3. A depiction of the collective coordinate representation. The degree of freedom $k(t)$ parametrizes the river valley, while the $w_a(t)$'s parametrize normal displacements

can be expanded to quadratic order in w_a:

$$Z \approx \int \mathcal{D}k(t)\mathcal{D}w_a(t)[\prod_t \|J(k(t), w_a(t))\|] \ \exp -\frac{1}{2e_0} \int dt\{[\int_0^L \beta_N(x,k)^2 \, dx] \ \dot{k}^2$$

$$+ \sum_a \dot{w}_a^2 + \sum_{ab} w_a \Omega^{ab}(k) w_b + source \ terms \ for \ w_a\} \ ,$$

where the matrix $\Omega(k)$ is the projection of the operator $-\frac{d^2}{dx^2}$ onto the subspace of normalizable functions satisfying Neumann boundary conditions and which are orthogonal to β_n. In other words

$$\Omega^{ab}(k) = \int_0^L \frac{dT^a(x)}{dx} \frac{dT^b(x)}{dx} \, dx \ .$$

The source term can be shown to be unimportant in the limit of infinite time evolution.

Only the first term in $\|J\|$ should be included to one loop. This factor cancels out if a change of variable is made from $k(t)$ to the arc-length parameter $\gamma(t)$ defined by

$$\frac{d\gamma}{dk} = \sqrt{\int_0^L \beta_N(x,k)^2 \, dx} \ .$$

We will write $k(t)$ as $k(\gamma(t))$ henceforth. For small k, $\gamma \approx \rho \approx k\sqrt{L}$.

The zero-point-energy contribution from the w_a's is simply half the sum of eigenfrequencies. The square of any one of these eigenfrequencies is an eigenvalue of $\Omega(k(\gamma))$. After integrating out these modes, we are left with

$$Z = \int \mathcal{D}\gamma(t) \exp - \int dt \; [\; \frac{\dot{\gamma}(t)^2}{2e_0} + \frac{1}{2} tr \sqrt{\Omega(k(\gamma(t)))} - \frac{1}{2} tr \sqrt{\Omega(0)} \;] \;.$$

Computing the eigenvalues of $\Omega(k)$ is done in the following way. The operator

$$\omega(k) = (1 - |\hat{\beta}_N > < \hat{\beta}_N|) \; (-\frac{d^2}{dx^2}) \; (1 - |\hat{\beta}_N > < \hat{\beta}_N|)$$

on the Hilbert space on $[0, L]$ with Neumann boundary conditions has the same eigenvalues as $\Omega(k)$. Here we use standard Dirac bra-ket notation for Hilbert space vectors. The eigenvalue problem for this operator is straightforward (we only present a quick and dirty derivation of the answer here).

Let $G(\lambda)$ be the inverse of $-d^2/dx^2 - \lambda$ on the Hilbert space. The operator $G(\lambda)$ isn't regular, since it has poles if $\lambda = n^2\pi^2/L^2$, but $G(\lambda)\sin\lambda$ is regular. Consider $G(\lambda)\sin\lambda|\hat{\beta}_N >$. If this is is orthogonal to $|\hat{\beta}_N >$, it must be an eigenvector of $\omega(k)$ with eigenvalue λ. By doing the analysis more carefully one can show that all the eigenvectors are of this form. The condition that λ be an eigenvalue of $\Omega(k)$ is therefore $\sin\lambda < \hat{\beta}_N|G(\lambda)|\hat{\beta}_N >= 0$ or

$$\int_0^L dx \int_0^{L-x} dy \; \cos\lambda x \cos\lambda y \; \hat{\beta}_N(x)\hat{\beta}_N(y) = 0 \;.$$

This function can be computed numerically, the zeros can be found, and the sum of the square roots obtained. A graph of the potential versus k (not γ) has vanishing slope at the origin and rises significantly only for $k \approx k^*$. One can therefore conclude that the gap to the first excited state is of order $1/L$.

Even without doing a very explicit calculation one can see that gap is impossible by a simple scaling argument. Each eigenvalue is directly proportional to $1/L^2$ and the sum of square roots of eigenvalues is finite (after making the subtraction at $k = 0$. Therefore, this sum must have the form

$$\frac{1}{2} tr \sqrt{\Omega(k(\gamma))} = \frac{1}{L} f(k(\gamma)) \;.$$

The only way any nontrivial γ dependence could emerge in the thermodynamic limit is if there is a term in $f(k)$ proportional to $1/k^2$, for small k. But then the result at large L would be

$$\frac{1}{2} tr \sqrt{\Omega(k(\gamma))} = \frac{C}{\gamma^2} \;,$$

where C is a constant for $\gamma < \sqrt{L}$. But this is a Calogero potential which has a continuous spectrum (there is no harmonic oscillator term).

THE $O(3)$ CASE AND THE LAMÉ EQUATION

The $O(3)$ model has two angles $\theta_1 = \theta$ and $\theta_2 = \phi$. Let us consider the lattice path integral

$$Z = [\prod_{x,t} \int_{-\pi/2}^{\pi/2} d\theta(x,t) \, |\sin\theta(x,t)| \,][\prod_{x,t} \int_{-\pi}^{\pi} d\phi(x,t)]$$

$$\times \, \exp - \sum_{x,t} \frac{1}{2e_0}\{[1 - \cos(\theta(x,t+a) - \theta(x,t))] + [1 - \cos(\theta(x+a,t) - \theta(x,t))]$$

$$+ \sin^2 \frac{\theta(x,t+a) + \theta(x,t)}{2} \, [1 - \cos(\phi(x,t+a) - \phi(x,t))]$$

$$+ \sin^2 \frac{\theta(x+a,t) + \theta(x,t)}{2} \, [1 - \cos(\phi(x+a,t) - \phi(x,t))]\} \, . \tag{3}$$

Here x and t are discrete, namely integers times the lattice spacing a (which is assumed to be much smaller than L/N). To calculate the zero-point energy of the fluctuations to one loop, the quantities $\sin^2(\theta(x+a,t) + \theta(x,t))/2$ in the action can be replaced by $\sin^2 \alpha_N$. Similarly the measure factor $\prod_{x,t} \sin|\theta(x,t)|$ can be replaced by $\prod_{x,t} \sin|\alpha_N(x,k(t))|$. Notice that $\sin^2 \alpha_N$ vanishes at $x_j = \frac{(2j-1)L}{2N}$, $j = 1,...,N$. This means that Neumann boundary conditions $\partial_x \phi = 0$ arise at these points; furthermore ϕ can be discontinuous at x_j. The degrees of freedom in the field ϕ do not communicate with one another across the line $x = x_j$. This breaking of the part of the action depending on ϕ into independent pieces in strips separated by the x_j is an artifact of the one-loop approximation. At higher loops, we can no longer assume the coefficients of the ϕ lattice derivatives vanish at these points.

As in the $O(2)$ case, vortex configurations of the θ field are of no importance at weak coupling. The contribution the fluctuations of this field give when integrated out is the same as before.

Consider next the integration over ϕ. If $\gamma << L$ (i.e. $|k| << 1$), then the coefficient of the ϕ lattice derivatives, namely $\sin^2 \alpha_N(x, k(t))$ is small over most of spacetime for small k and large over most of spacetime for large k. This function is essentially a slowly-varying inverse coupling constant for ϕ. We therefore expect ϕ-field vortices to be important, and we cannot treat the integral as a Gaussian in ϕ. However, let us come back to this point a little later and see what happens if the Gaussian approximation is used for the ϕ integration.

If we assume that $k(t)$ is slowly varying and ignore its time derivatives, we can absorb the measure factor $[\prod_{x,t} \sin \alpha_N(x, k(t))]$ into $\mathcal{D}\phi(x,t)$, by defining $\Phi = \phi \sin \alpha_N$ and take the continuum limit, obtaining

$$Z = \int \mathcal{D}\theta(x,t) \int \mathcal{D}\Phi(x,t) \exp - \int dt \int_0^L dx \, \frac{1}{2e_0}\{\partial_t \theta^2 + \partial_x \theta^2$$

$$+ \, [\partial_t\Phi^2 + \sin^2\alpha_N \, (\partial_x \frac{\Phi}{\sin\alpha_N})^2]\}$$

$$= \int \mathcal{D}\theta(x,t) \int \mathcal{D}\Phi(x,t) \exp - \int dt \int_0^L dx \, \frac{1}{2e_0}[\partial_t\theta^2 + \partial_x\theta^2 \qquad (4)$$

$$+ \, \Phi \, (-\partial_x^2 + \frac{24N^2K^2}{L^2}k^2 sn^2(u) - \frac{4N^2K^2}{L^2}(1+4k^2) \,) \, \Phi] \, ,$$

supplemented by Neumann boundary conditions $\partial_x\Phi = 0$ at $x = 0, L$ and Dirichlet boundary conditions $\Phi = 0$ at the points x_j.

After making the transformation from θ to k, w_1, w_2,... and integrating out both the w_a's and Φ gives

$$Z = \int \mathcal{D}\gamma(t) \exp - \int dt \, [\, \frac{\dot{\gamma}(t)^2}{2e_0} + \frac{1}{2}tr\sqrt{\Omega(k)} - \frac{1}{2}tr\sqrt{\Omega(0)}$$

$$+ \, \frac{2NK}{L}tr\sqrt{\mathcal{L}(k) + (1+4k^2)} - \frac{N\pi}{L}tr\sqrt{\mathcal{L}(0) + 1} \,] \qquad (5)$$

where Ω is defined as before and $\mathcal{L}$ is the Lamé operator

$$\mathcal{L}(k) = -\frac{d^2}{du^2} + m(m+1)k^2 sn^2(u) \, ,$$

where $m = 2$ in our case.

The eigenvalue problem $\mathcal{L}\Lambda(u) = \mathcal{A}\Lambda(u)$ is called the Lamé equation and was first solved in some generality by Hermite (see the book by Whittaker and Watson [19]).

A rough argument shows that the Gaussian approximation for ϕ does not yield a gap between the ground state and the first excited state. Large eigenvalues of the Lamé operator are well approximated by the eigenvalues of the Schrödinger operator with zero potential. The fourth term in the exponent of (5) is then an ultraviolet-divergent expression of the form

$$S(\Lambda, L, k) = \frac{A}{L} \sum_{n=1}^{\Lambda L} \sqrt{n^2 + f(k)} + o(\frac{1}{L}) \, ,$$

where A is a constant and $f(k)$ is some function with no x-dependence. The reason the mode sum is cut off at ΛL is because that is the number of degrees of freedom in the problem (for example, on a lattice, where Λ is the inverse lattice spacing). For large ΛL, the sum becomes an integral which can be evaluated to be

$$S(\Lambda, L, k) = \frac{A f(k)^2}{2L}[\sinh^{-1}\frac{\Lambda L}{f(k)} + \frac{1}{2}\sinh(2\sinh^{-1}\frac{\Lambda L}{f(k)}] + o(\frac{1}{L}) \, .$$

For large L all that remains is $S(\Lambda, L, k) \approx \frac{A\Lambda^2}{2}$ which has no k-dependence.

STRONG/WEAK-COUPLING DUALITY

Let us now look once again at (3). We will make a Gaussian approximation for θ, which we know to be justified, but not ϕ (we can differentiate between θ and ϕ using

the arguments in section 6). In order to find the effective action for γ, we need to find the contribution to the potential which is the free energy of the ϕ field with θ set equal to α_N, i.e.

$$W(\gamma) = -\lim_{T\to\infty} \frac{\log Z_\phi(\gamma)}{T} \ ,$$

where T is the time duration,

$$Z_\phi(\gamma) = [\prod_{x,t} \int_{-\pi}^{\pi} d\phi(x,t)\beta(x,\gamma)^{\frac{1}{2}}]$$
$$\times \ \exp -\sum_{x,t} \beta(x,\gamma)\{[1 - \cos(\phi(x,t+1) - \phi(x,t))] \tag{6}$$
$$+ [1 - \cos(\phi(x+1,t) - \phi(x,t))]\} \ ,$$

and

$$\beta(x,\gamma) = \frac{1}{2e_0} \sin^2 \alpha_N(x, k(\gamma)) = \frac{2k(\gamma)^2}{e_0} \ sn^2(u, k(\gamma)) \ dn^2(u, k(\gamma)),$$

plays the role of inverse coupling constant.

What is very striking is that no matter how small the coupling e_o may be, the effective coupling in the "ϕ sector" is large for $\gamma = k^2\sqrt{L} << \sqrt{L}$. This is a kind of strong-coupling/weak-coupling duality. It tells us that to study $W(\gamma)$ at weak coupling, we need a strong-coupling expansion. If there is a minimum of $W(\gamma)$ for finite γ, compactness effects, i.e. *vortices* are responsible. How can this be reconciled with the philosophy of our approximation, namely that only configurations close to the river valleys may be considered? The answer is that, unlike the case of the $O(2)$ sigma model, vortices are *short* paths in configuration space, whose lengths are not divergent. This point will be discussed in a later publication.

We have not yet proved the existence of a gap from the ground state to the first excited state, but it seems clear how the proof should go. First, the strong-coupling expansion will yield the potential $W(\gamma)$ (we have already found this). Then it must be checked that the gap does not disappear as $L \to \infty$. If this is so, the spatial correlation functions must automatically fall off exponentially; for if the wave function is localized at small k, the effective coupling of the ϕ-field must be strong. This is an important check of Lorentz invariance. Finally the dependence of the gap on e_0 must be checked for consistency with asymptotic freedom.

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